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Green functions for the Dirac equation for general classical solutions of the CP^{n-1} model

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Abstract. Green functions are constructed for the Dirac operator in two dimensions corresponding to the general solutions of the CP^{n-1} model in two dimensions recently obtained and analysed by Din and Zakrzewski. This is achieved both for the basic Dirac operator and also for that appropriate to the extended supersymmetric CP^{n-1} model. A brief discussion is given of the negative modes for the quadratic fluctuations about the classical action.

The two-dimensional CP^{n-1} model has proved an interesting field theoretic laboratory possessing many of the crucial features now considered significant in realistic four-dimensional gauge theories, but in a more tractable form. In particular, there is a topological charge taking integer values, and the Euclidean action for the model has minima for solutions of certain first-order equations, so that the action becomes proportional to the topological charge. These solutions correspond closely to the self- or anti-self-dual instanton solutions of Euclidean gauge theories. The contribution of these CP^{n-1} instantons to the functional integral defining the quantum field theory has been evaluated by Berg and Luscher (1980). A necessary part of calculating the functional determinants required in this calculation is to find Green functions for the relevant operators.

However, for the CP^{n-1} model with $n \geqslant 2$ the multi-instanton field configurations do not exhaust all possible solutions of the second-order classical equations obtained by requiring the action to be stationary. Din and Zakrzewski (1980a, b) have obtained a complete set of solutions of the nonlinear equations, with finite action, which, apart from the instantons or anti-instantons, are no longer minima but saddle points of the action.

In this paper we exhibit the associated Green functions for the Dirac operator for all such solutions, both in the case when the Dirac equation has essentially one component and in the supersymmetric version where there are n components. To establish notation and set the context for our investigation we briefly recapitulate and slightly rephrase some of the results of Din and Zakrzewski (1980a, b). In the CP^{n-1} model the basic field $z(x) \in C^n$ is subject to the constraint $\bar{z}z = |z|^2 = 1$ where the action is defined

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on equivalence classes $z(x) \sim e^{i\Lambda(x)} z(x)$ (resulting in a U(1) gauge invariance) corresponding to elements of CP^{n-1},

$$I = \frac{1}{g^2} \int d^2x\, \mathscr{L}, \qquad \mathscr{L} = \overline{D_\mu z} D_\mu z = 2(|D_+ z|^2 + |D_- z|^2), \tag{1}$$

with $D_\mu = \partial_\mu - \bar{z}\partial_\mu z$, $x_\pm = x_1 \pm i x_2$, $D_\pm = \frac{1}{2}(D_1 \mp i D_2)$. For suitably regular fields $z(x)$ the topological charge

$$q = \frac{1}{2\pi} \int d^2x\, Q, \qquad Q = -i\varepsilon_{\mu\nu} \overline{D_\mu z} D_\nu z = 2(|D_+ z|^2 - |D_- z|^2) \tag{2}$$

takes integer values.

The analysis of Din and Zakrzewski (1980a, b) for a general solution z of the classical equations obtained by requiring I in (1) to be stationary and with I finite proceeds by showing that for some l, m, $D_- z, \ldots, D_-^l z$ and $D_+ z, \ldots, D_+^m z$ are linearly independent and span mutually orthogonal l, m-dimensional vector spaces H_l^-, H_m^+ which are also orthogonal to z. Without loss of generality, it is sufficient to take $l + m = n - 1$ so that for every x there is an orthogonal decomposition

$$C^n = H_l^- + H_m^+ + H_z \tag{3}$$

where H_z is the one-dimensional vector space containing z. The solution z can then be reconstructed by demonstrating the existence of an analytic vector $f(x_+) \in H_l^- + H_z$, defined by requiring $\bar{f} D_-^i z = (-1)^l \omega \delta^{il}$, $i = 0, \ldots, l$, $D_+ \omega = 0$, and then showing that $\partial_+^h f$, $h = 0, \ldots, l$, are also linearly independent and span $H_l^- + H_z$ with $\overline{\partial_+^h f} z = \omega \delta^{hl}$. The projection operator on the space H_l^- is then given by

$$P_l^- = \sum_{h,i=0}^{l-1} \partial_+^i f M_{ih}^{(l)-1} \overline{\partial_+^h f}, \qquad M_{hi}^{(l)} = \overline{\partial_+^h f} \partial_+^i f, \qquad h, i = 0, \ldots, l-1, \tag{4}$$

and thus z is recovered in terms of f by

$$z = \hat{z}/|\hat{z}|, \qquad \hat{z} = (1 - P_l^-)\, \partial_+^l f, \tag{5}$$

so that $\omega = |\hat{z}|$. For any analytic $f(x_+)$, z, defined according to (4) and (5), obeys the classical equations which are tantamount to

$$P_l^- D_+ z = 0, \qquad P_m^+ D_- z = 0, \tag{6}$$

where P_m^+ is the corresponding projection operator on H_m^+, $P_l^- + P_m^+ = P = 1 - z\bar{z}$ (an equally valid alternative is to construct an analytic vector $g(x_-) \in H_m^+ + H_z$, the two bases for the vector spaces being related by $\overline{\partial_+^i f} \partial_-^j g = (-1)^i \delta_{n-1,i+j}$, $i + j \leq n - 1$). The vector f is arbitrary up to $f \to \lambda f$ for $\lambda(x_+)$ a scalar, since this corresponds to a gauge change in z, so that f may be supposed to be polynomial in x_+ with no overall zeros and $f(x_+) = O(x_+^k)$ for $|x| \to \infty$. Assuming, as appears to be the generic case, that f, $\partial_+ f, \ldots, \partial_+^l f$ are still linearly independent for all finite x, then $\hat{z}$, as defined in (5), has no zeros and behaves asymptotically for $|x| \to \infty$ like

$$\hat{z}(x) = O(x_+^q), \qquad z(x) \sim e^{i\theta q} z_\infty, \qquad e^{i\theta} = x_+/|x|, \tag{7}$$

where $q = k - 2l$. Such behaviour for $z(x)$ is as required for the one-point compactification of R^2 to S^2 (if $\xi_\mu = x_\mu/|x|^2$ we may take $z(x) = e^{i\theta q} z_\infty(\xi)$ with $z_\infty(\xi)$ regular at $\xi = 0$). As given in (7), q is identical with the topological charge defined in (2), since, using $D_+|\hat{z}|=0$ and $D_-z=\partial_-\hat{z}/|\hat{z}|$ so long as $|\hat{z}|$ has no singularities,

$$D_+=|\hat{z}|\partial_+(1/|\hat{z}|), \qquad D_-=(1/|\hat{z}|)\partial_-|\hat{z}|. \tag{8}$$

Hence $Q=\partial^2 \ln|\hat{z}|$ and with (7) gives an identity in (2).

The Dirac equation in two dimensions can be reduced to two one-component equations written in the form

$$D_\pm \phi^\pm = 0. \tag{9}$$

Corresponding to the compactification of the point at infinity, the relevant asymptotic behaviour for $|x|\to\infty$ is

$$\phi^\pm(x)=\mathrm{O}(\mathrm{e}^{\mathrm{i}\theta q}/x_\mp) \tag{10}$$

(with $\xi_\pm = 1/x_\mp$ we may take $\phi^\pm(x)=(1/x_\mp)\,\mathrm{e}^{\mathrm{i}\theta q}\phi^\pm_\infty(\xi)$). The solutions of (9), with (10), were given by Din and Zakrzewski (1981). For $q\geq 0$ they are easily seen, from (7) and (8), to require

$$\phi^+=0, \qquad \phi^-_\alpha = g_\alpha/|\hat{z}|, \qquad \alpha=1,\ldots,q, \tag{11}$$

for $\{g_\alpha\}$ a set of q independent polynomials of degree $q-1$ in x_+.

The associated Green function for the operator D_- can be constructed in terms of $G_0(x,x')$ which satisfies for $q\geq 0$

$$D_-G_0(x,x')=\delta^2(x-x'), \tag{12a}$$

$$G_0(x,x')\overleftarrow{D}'_- = \delta^2(x-x')-\sum_\alpha \phi^-_\alpha(x)\lambda_\alpha(x'), \tag{12b}$$

where $\overleftarrow{D}_\mu = -\overleftarrow{\partial}_\mu - \bar{z}\partial_\mu z$. λ_α is unconstrained save for

$$\int \mathrm{d}^2x\, \lambda_\alpha \phi^-_\beta = \delta_{\alpha\beta}. \tag{13}$$

The required asymptotic behaviour of G_0 in x, x' is, consistent with (7) and (10),

$$G_0(x,x')=\mathrm{O}(\exp(\mathrm{i}\theta q)/x_+, \exp(-\mathrm{i}\theta' q)/x'_+). \tag{14}$$

Once G_0 is found satisfying (12) and (14), the Green function G^-_ρ orthogonal to the zero modes (11) is given by

$$G^-_\rho = (1-Z^-_\rho)G_0 \tag{15}$$

where Z^-_ρ is the functional projection operator on the q-dimensional space spanned by the zero modes $\{\phi^-_\alpha\}$,

$$Z^-_\rho(x,x')=\sum_{\alpha,\beta}\phi^-_\alpha(x)\eta^{-1}_{\alpha\beta}\overline{\phi^-_\beta(x')}\rho(x')^{1/2}, \qquad \eta_{\alpha\beta}=\int \mathrm{d}^2x\,\rho^{1/2}\overline{\phi^-_\alpha}\phi^-_\beta, \tag{16}$$

where ρ serves to define a scalar product on the zero modes and depends on the metric. In a conformally flat space $g_{\mu\nu}=\rho\delta_{\mu\nu}$, and for the standard metric on S^2, with a radius a, projected on to R^2, $\rho(x)^{1/2}=2/(1+|x|^2/a^2)$. It is then straightforward to see that, as a result of (12), G_ρ satisfies

$$D_-G^-_\rho(x,x')=\delta^2(x-x'), \qquad G^-_\rho(x,x')\overleftarrow{D}'_- = \delta^2(x-x')-Z^-_\rho(x,x'). \tag{17}$$

Manifestly the only dependence of G_ρ on the metric is through the zero mode projector Z^-_ρ in (15). Consequently the explicit form of the metric can be avoided in the initial construction of G_0. For the operator D_+ the Green function can obviously be formed from that for D_-,

$$G_\rho^+(x,x') = -\overline{G_\rho^-(x',x)}, \tag{18}$$

by virtue of (17) and since $\overline{j\bar{D}_-} = -D_+\bar{j}$. For $q \leqslant 0$, in which case $\phi^- = 0$ but there are then $-q\,\phi_\beta^+$, the construction can be simply modified.

From (8) G_0 can be constructed to solve (12*a*), in the form

$$G_0(x,x') = \frac{1}{|\hat{z}(x)|}\frac{\overline{z(x')}F(x_+,x'_+)}{\pi(x_+-x'_+)} \tag{19}$$

so long as $F(x_+,x'_+)$ has no singularities in x_+ and satisfies

$$(1-P_l^-(x))F(x_+,x_+) = \hat{z}(x) \tag{20}$$

since

$$\partial_-[1/(x_+-x'_+)] = \pi\delta^2(x-x'). \tag{21}$$

To satisfy (14) it is necessary that

$$F(x_+,x'_+) = \mathrm{O}(x_+^q), \qquad |x|\to\infty, \tag{22a}$$

$$(1-P_l^-(x'))F(x_+,x'_+) = \mathrm{O}(1), \qquad |x'|\to\infty. \tag{22b}$$

To achieve (22*b*) as well as (20) it is sufficient to take

$$F(x_+,x'_+) = \partial_+^l f(x) - \sum_{i=0}^{l-1} a_i\left(\frac{\partial_+^i f(x') - \Sigma_{s=0}^{l-i}(1/s!)(x'_+-x_+)^s\partial_+^{s+i}f(x)}{(x'_+-x_+)^{l-i}}\right). \tag{23}$$

The coefficients a_i are determined by the requirement (22*a*). If $f(x) = \Sigma_{j=0}^k \alpha_j x_+^{k-j}$, with $\{\alpha_j\}$ linearly independent, then all terms $\alpha_j x_+^{k-l-j}$, $j=0,\ldots,l-1$, on the RHS of (23) must be cancelled. It is an exercise in combinatorics to show that this is possible if

$$a_i = (-1)^{l-i}\frac{(k-i)!(k-2l)!l!}{(k-i-l)!(k-l)!i!}. \tag{24}$$

The result (19) with (23) and (24) simplifies in two extreme cases: for $l=0$, $q=k$, which corresponds to the pure instanton solution with $D_-z=0$, $F(x_+,x'_+) = f(x)$ and

$$G_0(x,x') = \overline{z(x')}z(x)/\pi(x_+-x'_+); \tag{25}$$

also for $k=2l$, $q=0$, $F(x_+,x'_+) = \partial_+^l f(x')$ and

$$G_0(x,x') = \frac{|\hat{z}(x')|}{|\hat{z}(x)|}\frac{1}{\pi(x_+-x'_+)}. \tag{26}$$

To verify that (19) obeys (12*b*) requires

$$\sum_\alpha \phi_\alpha^-(x)\lambda_\alpha(x') = \frac{1}{|\hat{z}(x)|}\frac{\overline{D'_+z(x')F(x_+,x'_+)}}{\pi(x_+-x'_+)}. \tag{27}$$

From (23) $P_m^+(x)F(x_+,x_+) = 0$ and the singularity on the RHS of (27) is cancelled so it is a linear combination of the zero modes $\{\phi_\alpha^-\}$ given in (11), as necessary. A consistency check is given by taking the limit $x'\to x$ in (27), using

$$F(x_+,x'_+) = \partial_+^l f(x) + c(x_+-x'_+)\partial_+^{l+1}f(x) + \mathrm{O}((x_+-x'_+)^2),$$

$$c = q/(l+1)(k-l), \tag{28}$$

so that

$$\sum_\alpha \phi_\alpha^- \lambda_\alpha = \frac{c}{\pi} |D_+ z|^2. \tag{29}$$

From (13) it is then possible to count the number of zero modes, since

$$\int d^2x \sum_\alpha \phi_\alpha^- \lambda_\alpha = \frac{c}{\pi} \int d^2x\, |D_+ z|^2 = q \tag{30}$$

which is confirmed by virtue of the result (Din and Zakrzewski 1980b)

$$\frac{1}{\pi} \int d^2x\, |D_+ z|^2 = \frac{1}{4\pi} \int d^2x\, \partial^2 \ln \det M^{(l+1)} = (l+1)(k-l). \tag{31}$$

For the supersymmetric version of the Dirac equation ϕ is replaced by ψ, where ψ has n components and is orthogonal to z, $\bar{z}\psi = 0$. Corresponding to (9), the actual equations for zero modes are

$$PD_\pm \psi^\pm = 0, \qquad P = 1 - z\bar{z}, \tag{32}$$

where $\psi^\pm$ is required to have the same asymptotic behaviour as $\phi^\pm$ in (10). The solutions were obtained by Din and Zakrzewski and can be expressed in the form

$$\psi_a^- = \frac{1}{|\hat{z}|} P_m^+ g_a^-, \qquad a = 1, \ldots, n_-, \qquad \psi_b^+ = |\hat{z}| P_l^- g_b^+, \qquad b = 1, \ldots, n_+, \tag{33}$$

for $g_a^-(x_+)$, $g_b^+(x_-)$ linearly independent vectors polynomial in x_+, x_- restricted by the required asymptotic behaviour for $|x| \to \infty$. Verification of the result (32) relies on the relations

$$\begin{aligned} \partial_- P_l^- &= -D_- z\bar{z}, \qquad & \partial_+ P_l^- &= -z\overline{D_- z}, \\ \partial_+ P_m^+ &= -D_+ z\bar{z}, \qquad & \partial_- P_m^+ &= -z\overline{D_+ z}. \end{aligned} \tag{34}$$

The corresponding Green functions can be determined in terms of $\tilde{G}_0(x, x')$, which obeys

$$\begin{aligned} PD_- \tilde{G}_0(x, x') &= P\delta^2(x - x') - \sum_b f_b(x) \psi_b^+(x'), \\ \tilde{G}_0(x, x') \tilde{\overleftarrow{D}}'_- P' &= P\delta^2(x - x') - \sum_a \psi_a^-(x) e_a(x'), \end{aligned} \tag{35}$$

where e_a, f_b are restricted solely by

$$\int d^2x\, e_a \psi_{a'}^- = \delta_{aa'}, \qquad \int d^2x\, \overline{\psi_b^+} f_{b'} = \delta_{bb'}. \tag{36}$$

The Green function for PD_- orthogonal to the zero modes is then given in a similar fashion to (15),

$$\tilde{G}_\rho^- = (1 - \tilde{Z}_\rho^-) \tilde{G}_0 (1 - \tilde{Z}_\rho^{++}), \tag{37}$$

so that, in analogy with (17),

$$PD_- \tilde{G}_\rho^- = 1 - \tilde{Z}_\rho^{++}, \qquad \tilde{G}_\rho^- \tilde{\overleftarrow{D}}'_- P' = 1 - \tilde{Z}_\rho^-. \tag{38}$$

$\tilde{Z}_\rho^-$, $\tilde{Z}_\rho^+$ are the functional projector operators on the spaces spanned by the zero modes

$\{\psi_a^-\}$, $\{\psi_b^+\}$ constructed exactly as in (16). Of course the Green function for PD_{+}, $\tilde{G}_p^+$, can also be obtained just as in (18).

To determine $\tilde{G}_0$ it is useful to consider the projections onto the spaces H_l^-, H_m^+. Thus if we assume $P_m^+ f_b = 0$, which is quite permissible, then from (34) and (35),

$$PD_- P_m^+(x)\tilde{G}_0(x, x') = P_m^+\delta^2(x-x'),$$
$$P_m^+(x)\tilde{G}_0(x, x')\tilde{D}'_- P' = P_m^+\delta^2(x-x') - \sum_a \psi_a^-(x)e_a(x'), \tag{39}$$

which has the same form now as (12). It is then not difficult to see, using (34), that we can take

$$P_m^+(x)\tilde{G}_0(x, x') = P_m^+(x)G_0(x, x')P_m^+(x'). \tag{40}$$

The exhibition of the required form of the zero modes in (39) depends on $P_m^+(x)z(x') \propto (x'_+ - x_+)$, and we can then show that for $x' \to x$

$$\sum_a \psi_a^- e_a = \frac{1}{\pi}\{cP_m^+|D_+z|^2 + D_+z\overline{D_+z}\}. \tag{41}$$

This allows a check of the counting of the zero modes, as in (30), from (36)

$$n_- = \int d^2x \sum_a e_a\psi_a^- = \frac{1}{\pi}(cm+1)\int d^2x\, |D_+z|^2 = nq + l(l+1) \tag{42}$$

for $q = k - 2l$, in accord with Din and Zakrzewski (1981).

It remains to determine $P_l^-(x)\tilde{G}_0(x, x')$, which from (34) and (35) can be seen to satisfy

$$PD_- P_l^-(x)\tilde{G}_0(x, x') = P_l^-\delta^2(x-x') - \sum_b f_b(x)\overline{\psi_b^+(x')}, \tag{43a}$$

$$P_l^-(x)\tilde{G}_0(x, x')\tilde{D}'_- P' = P_l^-\delta^2(x-x'). \tag{43b}$$

A solution for (43*b*) can be given as

$$P_l^-(x)\tilde{G}_0(x, x') = \frac{P_l^-(x)}{|\hat{z}(x)|}\frac{E(x_+, x'_+)}{\pi(x_+ - x'_+)}P_l^-(x')|\hat{z}(x')| \tag{44}$$

with $E(x_+, x'_+)$ a square matrix acting from $H_l^-(x')$ to $H_l^-(x)$, $E(x_+, x_+) = 1$. The specific form of E is fixed by the requirement that $\tilde{G}_0$ have an asymptotic behaviour exactly as G_0 in (14). If we let

$$E(x_+, x'_+)\partial_+^i f(x') = h_i(x_+, x'_+), \qquad i = 0, \ldots, l-1,$$
$$h_i(x_+, x_+) = \partial_+^i f(x), \tag{45}$$

then it is necessary that

$$h_i(x_+, x'_+) = O(x_+^q, x_+'^{k-i-q}), \qquad |x|, |x'| \to \infty \tag{46}$$

and of course $h_i(x_+, x'_+)$ has no singularities in x_+, x'_+. This condition is solved by

$$h_i(x_+, x'_+) = \sum_{r=0}^{2l-1} c_{i,r}(x'_+ - x_+)^r\partial_+^{i+r}f(x),$$
$$c_{i,r} = \binom{2l-i}{r}\frac{(k-i-r)!}{(k-i)!}. \tag{47}$$

The result is easily seen to be compatible with (43*a*), and for $x' \to x$

$$\sum_b f_b \overline{\psi_b^+} = \frac{c_{l-1,1}}{\pi} D_- z \overline{D_- z}, \qquad c_{l-1,1} = \frac{l+1}{k-l+1}. \tag{48}$$

As before in (42) the zero modes can be counted,

$$n_+ = \int d^2x \sum_b \overline{\psi_b^+} f_b = \frac{c_{l-1,1}}{\pi} \int d^2x \; |D_- z|^2 = l(l+1), \tag{49}$$

since as in (30)

$$\frac{1}{\pi} \int d^2x \; |D_- z|^2 = \frac{1}{4\pi} \int d^2x \; \partial^2 \ln \det M^{(l)} = l(k-l+1). \tag{50}$$

Once more we find agreement with Din and Zakrzewski (1981).

For $q = 0$, $k = 2l$ it is easily seen that $E \to 1$ in (44), whereas for pure instantons $l = 0$, $q = k$, $P_0^- = 0$ and from (40) we have

$$\tilde{G}_0(x, x') = P(x) G_0(x, x') P(x') \tag{51}$$

where G_0 is given as in (25).

The usefulness of the decomposition (3), which played a crucial role in our analysis, can also be seen in other contexts. If we consider fluctuations about a general solution of the classical equations z_c which makes the action stationary,

$$z = z_c + h - \tfrac{1}{2}|h|^2 z_c + O(h^3), \qquad \bar{z}_c h = 0, \tag{52}$$

then

$$I = I_c + F + O(h^3) \tag{53}$$

where F is quadratic in h. If we write $P_m^+ h = h_+$, $P_l^- h = h_-$, $h = h_+ + h_-$ and drop the subscript c, now

$$\begin{aligned} F = -\frac{4}{g^2} \int d^2x \; (&\bar{h}_+ D_+ P_m^+ D_- h_+ + \bar{h}_- D_- P_l^- D_+ h_- + |D_- z|^2 \bar{h}_+ h_+ \\ &+ |D_+ z|^2 \bar{h}_- h_- - \bar{h}_- D_- z \bar{h}_+ D_+ z - \overline{D_- z h_- D_+ z h_+}). \end{aligned} \tag{54}$$

To discuss the semiclassical approximation to the functional integral it is necessary to know the zero modes and negative modes of the quadratic form F. In this context it is perhaps interesting to note that if we set $h_+ = \psi_a^-$, $h_- = \psi_b^-$ for any solutions as in (33), then $F \leq 0$, with $F = 0$ only if $h_+ = \lambda D_+ z$ and $h_- = \lambda^* D_- z$ (note that $D_\pm z$ can be expressed as in (33)), by elementary application of Schwarz-type inequalities. Since for the fluctuations h we may take $h = O(e^{i\theta q})$ for $|x| \to \infty$ there are at least, for $l > 0$, $(nq + 2l(l+1) + n - 2)$ linearly independent complex negative modes of F.

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